Quart. J. R. Met. Soc. (1970), **96**, pp. 313-319

551.501.721 : 521.32 : 551.524.3 : 551.571.4

The derivation of Swinbank's long-wave radiation formula

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(Manuscript received 20 October 1969)

SUMMARY

It is shown that Swinbank's empirical formula for the long-wave radiation from clear skies can be derived from existing knowledge of atmospheric emission by taking account of the moderately strong correlation at most places between screen temperature and the amount of water vapour overhead.

1. INTRODUCTION

In his analysis of surface observations of the long-wave radiation from clear skies, Swinbank (1963) found them to be well represented by the formula :

$$R\,(\mathrm{mW\ cm^{-2}}) = 5{\cdot}31\ .\ 10^{-14}\ T^{6} \qquad (1)$$

in which T = screen temperature, °K. Arising from this work and the subsequent discussion (Swinbank 1964), it became apparent that two effects, in greater or less proportion, contribute to the explanation of this result. One – the 'opacity effect' – is expressed in terms of the hypothesis that the depth of the surface atmospheric layer necessary to contain sufficient water vapour to provide effectively full emission in the relevant wave bands may always be so shallow as to differ very little in temperature from the surface air temperature, with the consequence that the actual amount of water vapour overhead is irrelevant to the formulation. The second factor to be considered, as pointed out by Monteith (1964), is the correlation between air temperature and the amount of water vapour in the vertical : this might also account for the absence of explicit reference to water vapour in Eq. (1).

It is the purpose of this note to show that the second effect is the more important one and that a close approximation to Swinbank's formula can be derived without invoking a significantly greater opacity effect than that indicated by current knowledge of atmospheric emissivities.

2. THE TEMPERATURE COEFFICIENT OF ATMOSPHERIC EMISSIVITY

To explain Eq. (1) solely in terms of the opacity effect, as expressed above, would require the atmospheric emissivity ($R/(\sigma T^4)$) to increase in proportion to the square of the absolute temperature. An examination of evidence on the temperature coefficient of emissivity of the lower atmosphere can help therefore in deciding to what extent the opacity effect may operate.

The emissivity of water vapour varies with temperature as a result of two effects; firstly, the shifting of the black-body energy distribution relative to the bands in the water vapour spectrum and secondly, the changing integral intensity of the bands. That the first effect leads to an emissivity decreasing with increasing temperature may be seen by considering a simplification of the water vapour spectrum. As illustrated by Fig. 1 the simplest model to consider is one in which the atmosphere is taken to be opaque to long-wave radiation apart from a completely transparent window at 8-13 μ m. With this model the variation of emissivity with temperature is readily evaluated from Planck's radiation formula – a calculation facilitated by Table 129 of *Smithsonian meteorological tables* (6th edition). The resulting values are :

Temperature, °K	250	275	300	325
Emissivity, per cent	73·7	70·2	67·7	66·3

Over the range 280-300°K covered by Swinbank's observational data the emissivity varies as $T^{-0{\cdot}4}$.

The positive temperature coefficient associated with the second effect outweighs the above negative coefficient and theory indicates that the combined effect is a fairly small positive coefficient at normal temperature for the lower atmosphere. From the Elsasser and Culbertson (1960) atmospheric radiation tables as amended by Zdunkowski, Barth and Lombardo (1966), it is found

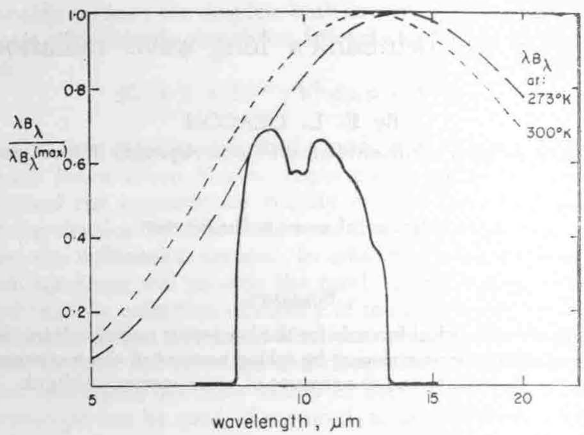

Figure 1. The water vapour window in relation to the black-body energy distributions for 273 °K and 300 °K. The spectrum reproduced with arbitrary ordinate scale from Roach and Goody (1958) is the difference between the emission from the spectometer and the emission from the sky on an occasion with 0·7 cm of precipitable water vapour in the zenith. λB_λ versus λ on a logarithmic scale gives areas proportional to energies.

that around 290°K the isothermal emissivity of water vapour increases approximately as $T^{0\cdot 2}$ with an optical depth of water vapour $u = 2$ cm : at $u = 0{\cdot}1$ cm the index is about 0·15.

The theoretical value of the temperature coefficient may be somewhat in error owing to the fact that the nature of the water vapour emission in the window region of the spectrum is still not well understood. However, the present indication from theory is that the 'opacity effect' can only play a minor role in explaining Swinbank's formula.

3. The variation of long-wave radiation with zenith angle

With an opaque-and-clear model of the atmosphere such as that considered above, there would be no variation of the radiation from the atmosphere with zenith angle. As the full 'opacity effect' explanation of Eq. (1) implies that the atmosphere approximates rather closely to such a model, it is of interest to examine how well existing theory predicts the variation with zenith angle which is actually observed.

In Fig. 2 some sets of observations are shown. For comparison with those of Dines and Dines (1927) for June at Benson, Oxfordshire, there is also given a calculated relationship (full line) on the basis of Elsasser's (1942) generalized absorption coefficients. A computer programme compiled by Dr. G. W. Paltridge for another study was available for this purpose and was applied to the mean vertical profiles of temperature and humidity up to 400 mb from Mildenhall ascents on 11 clear afternoons in June 1940. The agreement between theory and observation is satisfactory: the evidence is that an increasing amount of water vapour in the line of sight does appreciably increase the radiation. The radiation from two air masses is 7 to 8 per cent more than from one air mass, for the low level stations, and with the smaller amount of water vapour above Pikes Peak (elevation 4,300 m) the corresponding difference amounts to 10 per cent.

4. A derivation of Swinbank's formula

Elsasser (1942) pointed out that his results indicate that the isothermal emissivity of atmospheric water vapour and carbon dioxide varies nearly linearly with the logarithm of the optical depth of water vapour (u) as is consistent with the near linearity of the curves in Fig. 2. Neglecting the small temperature coefficient of emissivity gives the approximation that

$$R/(\sigma T^4) = a + b \log u \qquad (2)$$

Yamamoto (1952) on the basis of his radiation chart, which improves on Elsasser's treatment of the water vapour/carbon dioxide spectral overlap, found the following values of the constants

$$a = 0{\cdot}732, \quad b = 0{\cdot}165 \qquad (2a)$$

for u in cm as defined by Eq. (4) and logarithms to base 10.

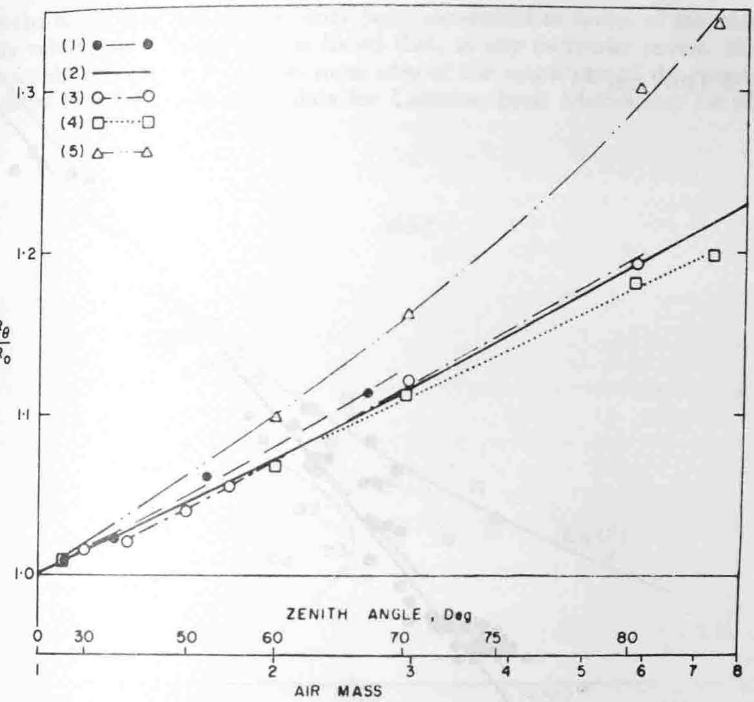

Figure 2. The variation of long-wave radiation with air mass on a logarithmic scale : clear sky condition.
(1) Observations of Dines and Dines (1927) for June at Benson, Oxfordshire ($u \sim 1{\cdot}9$ cm).
(2) Calculation from Mildenhall June ascent data.
(3) Observations of Kondrat'yev and Elovskikh (1955) at Voeikovo, near Leningrad : 18 sets of observations in June and July.
(4) Observations at Colorado Springs, Colorado } Bennett, Bennett and Nagel (1960).
(5) Observations at Pikes Peak (4,300 m), Colorado }

Now it has been shown elsewhere (Deacon 1963) that at many places near sea-level there is a reasonably close relationship between the amount of precipitable water (w) and mean air temperature at screen level. This arises from the fact that over the range 0-40°C the saturation vapour pressure of water is very nearly proportional to the 18th power of the absolute temperature. From monthly means at places covering a considerable climatic range the relationship obtained is

$$\log w = 16{\cdot}8 \log T - 40{\cdot}97 \quad . \quad . \quad . \quad . \quad (3)$$

in which the units are w cm and T °K. This differs but little from a formula proposed by Montefinale and Papee (1961) on the basis of radiosonde data for 15 stations at elevations less than 180 m in U.S.A.

To be able to combine Eq. (3) with Eq. (2) to arrive at an $R : T$ relationship one needs to be able to correct w to optical depth u by using

$$u = \int_0^\infty \rho_w \, (p/p_s) \, dz \quad . \quad . \quad . \quad . \quad (4)$$

in which ρ_w = density of water vapour and p, p_s are atmospheric pressures at height z and at standard surface pressure (1,000 mb) respectively. The mean profiles of specific humidity up to 300 mb given by Bannon and Steele (1960) for Larkhill, Wiltshire give $u/w = 0{\cdot}81$ for January and 0·80 for July and their world charts of w at various pressure levels show 0·80 to be a good approximation generally.

With $u/w = 0{\cdot}80$ Eqs. (2), (2a) and (3) give :

$$R = \sigma T^4 (2{\cdot}772 \log T - 6{\cdot}044) \quad . \quad . \quad . \quad (5)$$

Log R versus log T from Eq. (5) is shown plotted in Fig. 3 which is a reproduction of Swinbank's (1963) Fig. 3. The value of σ used is that employed by Swinbank i.e. $5{\cdot}77 \times 10^{-9}$ mW cm^{-2} K^{-4}.

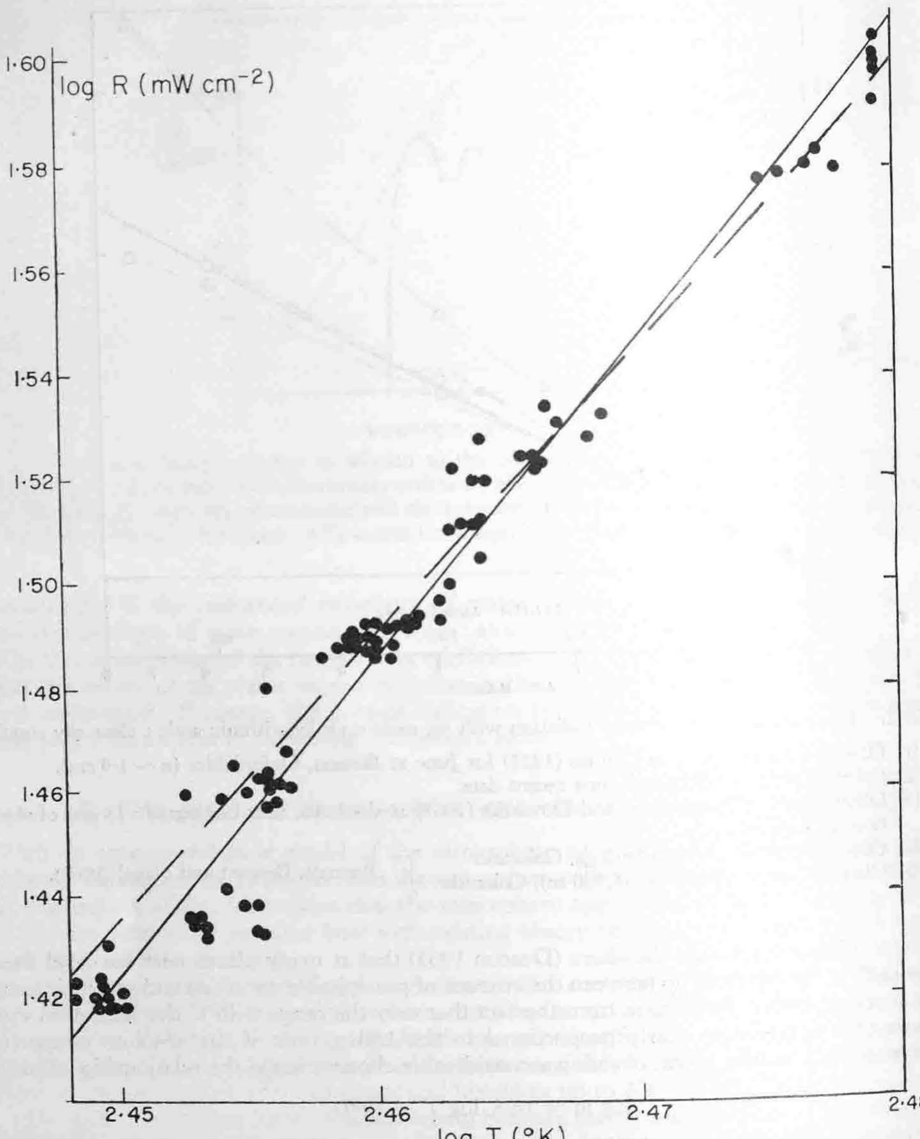

Figure 3. Swinbank's (1963) observations of R related to screen temperature T.

Full line – Swinbank's formula, Eq. (1).
Broken line – Eq. (5).

On the log : log plot of Fig. 3 Eq. (5) is almost linear and approximates closely to

$$R = 2{\cdot}49 \times 10^{-14}\, T^{5{\cdot}53}. \qquad (6)$$

Swinbank's empirical value of the index of T is $6{\cdot}15 \pm 0{\cdot}4$ which is not significantly different from that deduced here. There is perhaps some suggestion that the emissivity of water vapour may increase rather more rapidly with temperature than indicated by theory. Robinson (1950) also found some indication of this : however, to disentangle with precision this effect from that arising from the w : T correlation is likely to be a matter of difficulty requiring many observations in conjunction with ascent data at localities remote from aerosol pollution. Fairly detailed temperature profiles from the ground up to some 50 or 100 m height would also be needed, as is apparent from radiation chart plottings of typical clear sky occasions.

So far the $w : T$ relationship has only been considered in terms of monthly mean values. When daily values are considered it is found that, at any particular season, the correlation is smaller, as would be expected. To get some idea of the magnitude of the correlation, w-values were calculated from the radiosonde data for Laverton (near Melbourne) for two Junes. The

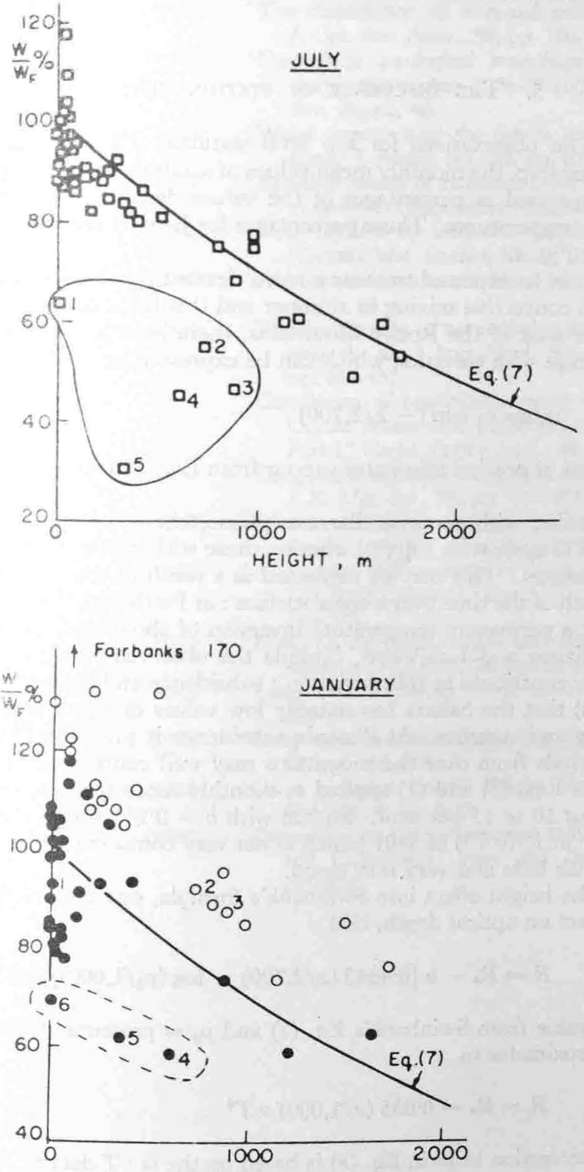

Figure 4. Amounts of precipitable water vapour (w) as percentages of that given by Eq. (3) in relation to height of station : 50 stations in U.S.A.

Open circles : Jan. mean temperature $\leqslant 0°C$.
Filled circles : Jan. mean temperature $> 0°C$.

1. Medford, Oregon
2. Spokane, Washington
3. Boise, Idaho
4. Las Vegas, Nevada
5. Phoenix, Arizona
6. San Diego, California

June data were chosen as the Laverton ascents at 09.00 h local time are then at approximately the time of zero lapse rate in the atmosphere near the ground. Surface to 700 mb w-values were used owing to the paucity of humidity values for higher levels. The correlation coefficient between w and T was 0·64. This is not significantly different from the correlation coefficient of 0·66 between w and surface vapour pressure – the quantity acting as a measure of w in several well known empirical long-wave radiation formulae.

5. The influence of station elevation

Eq. (3) is based on observations for low level stations. To examine the effect of height of station on the relationship, the monthly mean values of w tabulated by Reitan (1960) for stations in U.S.A. may be expressed as percentages of the values derived from Eq. (3) applied to the monthly mean screen temperatures. These percentages for January and July are shown in relation to station elevation in Fig. 4.

The July results can be expected to show a more clearcut relationship than those for January owing to the vigorous convective mixing in summer and this is borne out by Fig. 4 : apart from several stations to the west of the Rocky Mountains, there is in July a reasonably well defined decline of the percentage with elevation which can be expressed by

$$w/w_F = \exp(-z/2{,}700) \qquad (7)$$

in which w_F = amount of precipitable water vapour from Eq. (3) and z = height of station above m.s.l. in metres.

The January results, with some similar exceptions, show that places with January mean temperatures above 0°C agree with Eq. (7), whereas those with below freezing mean temperatures all give higher percentages. This may be explained as a result of the strong surface temperature inversion present much of the time over a snow surface : at Fairbanks, Alaska where w/w_F reaches 1·7 there is in winter a permanent temperature inversion of about 10°C (Byers 1940).

At Phoenix, Arizona and Las Vegas, Nevada the observed w-values are particularly low. Two factors probably contribute to this deviation : subsidence and föhn effect. It was previously found (Deacon 1963) that the Sahara has notably low values of w/w_F owing to subsidence – as low as 0·25 in spring and summer. At Phoenix subsidence is probably the main factor but the prevailing easterly winds from over the mountains may well contribute a föhn component.

For many places Eqs. (3) and (7) applied to monthly mean temperatures should give mean w-values within about 10 or 15 per cent. Eq. (2) with $b = 0{\cdot}165$ shows that a 15 per cent error in u causes an error in $R/(\sigma T^4)$ of 0·01 which is not very considerable in relation to the uncertainties associated with haze and very thin cloud.

Incorporating the height effect into Swinbank's formula, one obtains from Eqs. (2) and (7) and the pressure effect on optical depth, that

$$R = R_s - b\,\{0{\cdot}4343\,(z/2{,}700) - \log(p_0/1{,}000)\}\,\sigma T^4$$

in which R_s is the value from Swinbank's Eq. (1) and p_0 = pressure at station level, mb. This with $b = 0{\cdot}165$ approximates to

$$R = R_s - 0{\cdot}035\,(z/1{,}000)\,\sigma T^4 \qquad (8)$$

Although the correction term in Eq. (8) is based on the w : T data to no more than 1,700 m, it is of interest to see that it conforms reasonably well with Ångström's (1915) observations of R during 8 clear nights on Mt. Whitney (4,420 m). Without correction Swinbank's formula predicts values of $R/(\sigma T^4)$ from 8 to 29 per cent greater than observed, with a mean of 21 per cent. With the Eq. (8) correction included the mean difference is reduced to a 6 per cent underestimation.

Acknowledgment

The author is indebted to Dr. G. W. Paltridge for the angular distribution computations and for helpful suggestions.

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